Nonlinear filtering for jump diffusions

Measure-valued SPDEs and Sobolev densities

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Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete filtered probability space, $(W_t, V_t)_{t\geqslant 0}$ a $d_1 + d'$-dimensional $\mathcal{F}_t$-**Wiener process**, $\tilde{N}_i(d\mathfrak{z}, dt) = N_i(d\mathfrak{z}, dt) - \nu_i(d\mathfrak{z})dt$ independent $\mathcal{F}_t$-**Poisson martingale measures** on $\mathbb{R}_+ \times \mathfrak{Z}_i$, for $i = 0, 1$, with σ-finite characteristic measures ν_0 and ν_1 on separable measurable spaces $(\mathfrak{Z}_i, \mathcal{Z}_i)$ for $i = 0, 1$, where we assume $\mathfrak{Z}_1 = \mathbb{R}^{d'}$, $\mathcal{Z}_1 = \mathcal{B}(\mathbb{R}^{d'})$. We consider the signal and observation model

$$dX_t = b(t, Z_t)dt + \sigma(t, Z_t)dW_t + \rho(t, Z_t)dV_t + \int_{\mathfrak{Z}_0} \eta(t, Z_{t-}, \mathfrak{z})\, \tilde{N}_0(d\mathfrak{z}, dt) + \int_{\mathfrak{Z}_1} \xi(t, Z_{t-}, \mathfrak{z})\, \tilde{N}_1(d\mathfrak{z}, dt)$$

$$dY_t = B(t, Z_t)dt + dV_t + \int_{\mathfrak{Z}_1} \mathfrak{z}\, \tilde{N}_1(d\mathfrak{z}, dt),$$

for $Z_t = (X_t, Y_t) \in \mathbb{R}^{d+d'}$ and Borel measurable coefficients.

Assumptions

Let $K_0, K_1, L \geqslant 0$, $\kappa < 1$ and $\bar{\xi} \in L_2(\mathfrak{Z}_1)$, $\bar{\eta} \in L_2(\mathfrak{Z}_0)$, $\bar{\eta}, \bar{\xi} \leqslant K$.

(A1) $b, B, \sigma, \rho, (\sigma B)$ and η, ξ are **globally Lipschitz** on $\mathbb{R}^{d+d'}$, **uniformly in** t, with constants L and $\bar{\eta}(\mathfrak{z}), \bar{\xi}(\mathfrak{z})$.

(A2) For all $z \in \mathbb{R}^{d+d'}$, $t \geqslant 0$ and $\mathfrak{z}_i \in \mathfrak{Z}_i$, we have $|B| \leqslant K$ and

$$|b(t,z)|^2 + |\sigma(t,z)|^2 + |\rho(t,z)|^2 \leqslant K_0 + K_1|z|^2,$$

$$|\eta(t,z,\mathfrak{z}_0)|^2 \leqslant \bar{\eta}(\mathfrak{z}_0)(K_0 + K_1|z|^\kappa), \quad |\xi(t,z,\mathfrak{z}_1)| \leqslant \bar{\xi}^2(\mathfrak{z}_1)(K_0 + K_1|z|^\kappa).$$

(A3) The **initial condition** $Z_0 = (X_0, Y_0)$ satisfies $\mathbb{E}|Z_0|^2 < \infty$.

(M) Let for some $\varepsilon > 0$, the **measure** ν_1 satisfy

$$\int_{\mathfrak{Z}_1} |\mathfrak{z}|^{2+\varepsilon}\, \nu_1(d\mathfrak{z}) < \infty.$$

(D) The functions $f(\cdot) = \xi(t, \cdot, y, \mathfrak{z})$ and $f(\cdot) = \eta(t, \cdot, y, \mathfrak{z})$ are **continuously differentiable** for each $(t, y, \mathfrak{z}) \in \mathbb{R}_+ \times \mathbb{R}^{d'} \times \mathfrak{Z}_i$. Moreover, for a $\lambda > 0$ we have for all $\theta \in [0,1]$, $t \geqslant 0$,

$$|D_x f(t,x,y,\mathfrak{z})| \leqslant \lambda, \quad \text{and} \quad |\det(\mathbb{I} + \theta D_x f(t,x,y,\mathfrak{z}))| \geqslant \lambda^{-1}.$$

Some notation

Define the **processes**

$$\gamma_t = \exp\left(-\int_0^t B(s, X_s, Y_s)\, dV_s - \tfrac{1}{2}\int_0^t |B(s, X_s, Y_s)|^2\, ds\right),$$

$$\tilde{V}_t := \int_0^t B(s, Z_s)\, ds + V_t, \quad a_t^{ij} := \tfrac{1}{2}\sum_{k=1}^{d_1}(\sigma_t^{ik}\sigma_t^{jk}) + \tfrac{1}{2}\sum_{l=1}^{d_2}(\rho^{il}\rho_t^{jl}),$$

the measure $Q = \gamma_T P$ and the **differential operators**

$$\mathcal{L}_t = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i, \quad \mathcal{M}_t^k = \rho_t^{ik}(x)D_i + B_t^k(x)$$

$$I_t^\xi \varphi(x, \mathfrak{z}) = \varphi(x + \xi_t(x, \mathfrak{z}), \mathfrak{z}) - \varphi(x, \mathfrak{z})$$

$$J_t^\xi \phi(x, \mathfrak{z}) = I_t^\xi \phi(x, \mathfrak{z}) - \sum_{i=1}^d \xi_t^i(x, \mathfrak{z}) D_i \phi(x, \mathfrak{z}).$$

for functions $\varphi(x, \mathfrak{z})$ and $\phi(x, \mathfrak{z})$ of $x \in \mathbb{R}^d$ and $\mathfrak{z}_i \in \mathfrak{Z}_i$.

The **objective of filtering** is to **characterize** $\mathbb{E}(\varphi(X_t)|\mathcal{F}_t^Y)$ for a class of φ, where the *observation filtration*

$$\mathcal{F}_t^Y = \sigma(\{Y_s : s \in [0,t]\}).$$

Theorem I: the filtering equations

Let (A1)-(A3) hold. Then there exist **measure-valued** $\mathcal{F}_t^Y$-adapted weakly càdlàg **processes** $(P_t)_{t\in[0,T]}$ and $(\mu_t)_{t\in[0,T]}$ such that (a.s.) for each $t \in [0,T]$

$$P_t(\varphi) = \mu_t(\varphi)/\mu_t(\mathbf{1}),$$

$$P_t(\varphi) = \mathbb{E}(\varphi(X_t)|\mathcal{F}_t^Y), \quad \mu_t(\varphi) = \mathbb{E}_Q(\gamma_t^{-1}\varphi(X_t)|\mathcal{F}_t^Y)$$

for bounded φ on $\mathbb{R}^d$, and for every $\varphi \in C_b^2(\mathbb{R}^d)$ a.s.

$$\begin{aligned}\mu_t(\varphi) =& \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{L}_s\varphi)\, ds + \int_0^t \mu_s(\mathcal{M}_s^k\varphi)\, d\tilde{V}_s^k \\ &+ \int_0^t\int_{\mathfrak{Z}_0} \mu_s(J_s^\eta\varphi)\, \nu_0(d\mathfrak{z})ds + \int_0^t\int_{\mathfrak{Z}_1} \mu_s(J_s^\xi\varphi)\, \nu_1(d\mathfrak{z})ds \\ &+ \int_0^t\int_{\mathfrak{Z}_1} \mu_s(I_s^\xi\varphi)\, \tilde{N}_1(d\mathfrak{z}, ds),\end{aligned}$$

$$\begin{aligned}P_t(\varphi) =& P_0(\varphi) + \int_0^t P_s(\mathcal{L}_s\varphi)\, ds + \int_0^t\int_{\mathfrak{Z}_1} P_{s-}(I_s^\xi\varphi)\, \tilde{N}_1(d\mathfrak{z}, ds) \\ &+ \int_0^t\int_{\mathfrak{Z}_0} P_s(J_s^\eta\varphi)\, \nu_0(d\mathfrak{z})ds + \int_0^t\int_{\mathfrak{Z}_1} P_s(J_s^\xi\varphi)\, \nu_1(d\mathfrak{z})ds \\ &+ \int_0^t (P_s(\mathcal{M}_s^k\varphi) - P_s(\varphi)P_s(B_s^k))\, d\bar{V}_s^k\end{aligned}$$

for all $t \in [0,T]$, where

$$d\bar{V}_t := d\tilde{V}_t - P_t(B_t)\, dt = dV_t + (B_t(X_t) - P_t(B_t))\, dt, \quad \bar{V}_0 = 0.$$

Theorem II: L_p-valued densities

Let (A1)-(D) hold for some $\varepsilon > 0$ and assume $E|X_0|^{2+\varepsilon} < \infty$. Let moreover the initial conditional density $\pi_0 = dP(X_0 \in dx|Y_0)/dx$ exist and $\mathbb{E}|\pi_0|_{L_p}^p < \infty$ for some $p \geqslant 2$. Then almost surely the conditional density $\pi_t := P(X_t \in dx|\mathcal{F}_t^Y)/dx$ exists for all $t \in [0,T]$. Moreover, $(\pi_t)_{t\in[0,T]}$ **is an** L_p**-valued weakly cadlag process**.

Theorem III: W_p^m-valued densities

Let the conditions of Theorem II hold. Let $m > 0$ and assume the **coefficients** b, B, σ, ρ and η, ξ admit $[m] + 1$ **derivatives** in $z \in \mathbb{R}^{d+d'}$, bounded by L and $L\bar{\eta}, L\bar{\xi}$ respectively. $p \geqslant 2$ and $\mathbb{E}|\pi_0|_{W_p^m}^p < \infty$.

Then almost surely $\pi_t = dP(X_t \in dx|\mathcal{F}_t^Y)/dx$ exists and **is a weakly càdlàg** W_p^m**-valued process**.

1. F. Germ and I. Gyöngy, *On conditional densities of jump diffusions I: the filtering equations*, in preparation
2. F. Germ and I. Gyöngy, *On conditional densities of jump diffusions II: integrability of the filter*, in preparation
3. F. Germ and I. Gyöngy, *On conditional densities of jump diffusions III: regularity of the filter*, work in progress